A note on the amplitude modulation of symmetric regularized long-wave equation with quartic nonlinearity

Hilmi Demiray

e-mail:demiray@isikun.edu.tr

Department of Mathematics, Isik University, 34980 Sile-Istanbul-Turkey

Abstract We study the amplitude modulation of a symmetric regularized long-wave equation with quartic nonlinearity through the use of the reductive perturbation method by introducing a new set of slow variables. The nonlinear Schrödinger (NLS) equation with seventh order of nonlinearity is obtained as the evolution equation for the lowest order term in the perturbation expansion. It is also shown that the NLS equation with seventh order of nonlinearity assumes an envelope type of solitary wave solution.

1 Introduction

As is well known, when the nonlinear effects are small, the system of linear equations that describe the physical phenomenon admit harmonic wave solution with constant amplitude. If the amplitude of the wave is small- but - finite, e.g., weakly nonlinear, the nonlinear terms cannot be neglected and the nonlinearity gives rise to the variation of amplitude both in space and time variables. When the amplitude varies slowly over a period of oscillation, a stretching transformation allows us to decompose the system into a rapidly varying part associated with the oscillation and a slowly varying part for the amplitude. A formal solution can be given in the form of an asymptotic expansion, and an equation determining the modulation of the first order amplitude can be derived. For instance, the nonlinear Schrödinger(NLS) equation is the simplest representative equation describing the self-modulation of one dimensional monochromatic plane waves in dispersive media. It exhibits a balance between the nonlinearity and dispersion.

Due to its central importance to the theory of quantum mechanics, the nonlinear equation of Schrödinger type has a great interest. They arise in many nonlinear physical problems such as water waves [1-6], waves in plasma [7-11], nonlinear waves in a fluid-filled elastic or viscoelastic tubes [12-16] and other nonlinear waves of similar nature. In all these works only the effects of quadratic or cubic nonlinearities have been taken into account.

In studying the amplitude modulation of nonlinear partial differential equation, if the order of nonlinearity is two (quadratic) or three (cubic) it is a standard technique to introduce the stretched coordinates $\xi = \epsilon(x - \lambda t)$, $\tau = \epsilon^2 t$, where ϵ is the smallness parameter for the band width of the wave packet and λ is a parameter which is shown to be group velocity of the linearized harmonic wave. In obtaining the corresponding evolution equation for the nonlinear partial differential equation the field variables are assumed to be functions of the slow variables $(\xi,\ \tau)$ as well as the fast variables $(x,\ t)$. By use of the conventional reductive perturbation method it can be shown that the evolution equation for the lowest order term in the perturbation expansion will be the nonlinear Schrödinger equation [14, 16]. However, when the order of nonlinearity is four or higher, the use of the classical perturbation method does not lead to the nonlinear Schrödinger (NLS) equation, it rather leads to the degenerate (linearized) form of the NLS equation. In order to obtain the nonlinear Schrödinger equation, a new set of slow variables must be introduced.

In the present work, we study the amplitude modulation of a symmetric regularized long-wave equation with quartic nonlinearity through the use of the reductive perturbation method by introducing a new set of slow variables. The nonlinear Schrödinger (NLS) equation with seventh order of nonlinearity is obtained as the evolution equation for the lowest order perturbation expansion. Considering the sign of the product of some coefficients a progressive wave type of solution to the evolution equation is presented. It is shown that the NLS equation with seventh order of nonlinearity also assumes an envelope type of solitary wave solution.

2 Formulation of the problem

The regularized long-wave (RLW) equation has been the focus of appreciable attention primarily because of the intriguing numerical properties associated with the inelastic scattering of solitary waves [17]. The RLW equation is

$$u_t + u_x - uu_x - u_{xxt} = 0, \tag{1}$$

where $u(x,t)$ is the fluid velocity in x direction, x and t are the space and time variables, respectively. This equation possesses the solitary wave solution of the form $u = a\ \mathrm{sech}^2[p(x - vt)],\ a = 3(1-v),\ p = [(v-1)/4v]^{1/2}$. Starting with the equations of cold-electron plasma, for a weakly nonlinear case and in the long-wave approximation, Seyler and Fenstermacher [18] obtained the following evolution equation

$$u_{tt} - u_{xx} + (\frac{1}{2}u^2)_{xt} - u_{xxtt} = 0, \tag{2}$$

Due to its similarity to the RLW equation and the explicit symmetry in the derivatives of x and t, the authors named it as the symmetric regularized long-wave (SRLW) equation.

Chen [19], and Yong and Biao [20] extended this equation for a function $f(u)$ of class C^1 as

$$u_{tt} - u_{xx} + (f(u))_{xt} - u_{xxtt} = 0, \tag{3}$$

and named it as the generalized symmetric regularized long-wave (gSRLW)equation. In this work we will be concerned with a function of the form $f(u) = u^4/4$. Then, the gSRLW equation becomes

$$u_{tt} - u_{xx} + \frac{1}{4}(u^4)_{xt} - u_{xxtt} = 0. \tag{4}$$

Seeking a harmonic wave solution, $u = u_0 \exp[i(\omega t - kx)]$, to the linearized part of equation (4), i.e.,

$$u_{tt} - u_{xx} - u_{xxtt} = 0, \tag{5}$$

the dispersion relation is obtained as

$$\omega^2(1 + k^2) - k^2 = 0, \tag{6}$$

where ω is the frequency and k is the wave number of the harmonic wave.

Motivated with the dispersion relation (6) we shall introduce the following slow variables

$$\xi = \epsilon^3(x - \lambda t), \quad \tau = \epsilon^6 t, \tag{7}$$

where ϵ ($\epsilon^3 = \Delta k$) is the smallness parameter measuring the band width (Δk) of the wave packet and λ is a scale parameter which will be shown to be the group velocity.

We shall assume that the field variable u is a function of the slow variables (ξ, τ) as well as the fast variables (x, t). Thus, the following operators are valid

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \epsilon^3\lambda\frac{\partial}{\partial \xi} + \epsilon^6\frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon^3\frac{\partial}{\partial \xi}. \tag{8}$$

We shall further assume that the field variable u can be expanded into a perturbation series in ϵ as

$$u = \epsilon u_1 + \epsilon^4 u_2 + \epsilon^7 u_3 + \epsilon^{10} u_4 + ... \quad . \tag{9}$$

Introducing the expansions (8) and (9) into equations (4) and setting the coefficients of like powers of ϵ equal to zero the following sets of differential equations is obtainrd:

$O(\epsilon)$ equation:

$$\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^4 u_1}{\partial x^2 \partial t^2} = 0. \tag{10}$$

$O(\epsilon^4)$ equation:

$$\begin{aligned} &\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^4 u_2}{\partial x^2 \partial t^2} - 2\left(\frac{\partial^2 u_1}{\partial x \partial \xi} + \lambda \frac{\partial^2 u_1}{\partial t \partial \xi}\right) \\ &-2\left(\frac{\partial^4 u_1}{\partial x \partial t^2 \partial \xi} - \lambda \frac{\partial^4 u_1}{\partial x^2 \partial t^2 \partial \xi}\right) + \frac{1}{4}\frac{\partial^2}{\partial x \partial t}(u_1^4) = 0. \end{aligned} \tag{11}$$

$O(\epsilon^7)$ equation:

$$\begin{aligned} &\frac{\partial^2 u_3}{\partial t^2} - \frac{\partial^2 u_3}{\partial x^2} - \frac{\partial^4 u_3}{\partial x^2 \partial t^2} - 2\left(\frac{\partial^2 u_2}{\partial x \partial \xi} + \lambda \frac{\partial^2 u_2}{\partial t \partial \xi}\right) \\ &-2\left(\frac{\partial^4 u_2}{\partial x \partial t^2 \partial \xi} - \lambda \frac{\partial^4 u_2}{\partial x^2 \partial t \partial \xi}\right) + (\lambda^2 - 1)\frac{\partial^2 u_1}{\partial \xi^2} + 2\frac{\partial^2 u_1}{\partial t \partial \tau} \\ &+\frac{\partial^2}{\partial x \partial t}(u_1^3 u_2) - 2\frac{\partial^4 u_1}{\partial x^2 \partial t \partial \tau} + 4\lambda \frac{\partial^4 u_1}{\partial x \partial t \partial \xi^2} \\ &-\frac{\partial^4 u_1}{\partial t^2 \partial \xi^2} - \lambda^2 \frac{\partial^4 u_1}{\partial x^2 \partial \xi^2} + \frac{1}{4}\frac{\partial^2}{\partial t \partial \xi}(u_1^4) - \frac{\lambda}{4}\frac{\partial^2}{\partial x \partial \xi}(u_1^4) = 0. \end{aligned} \tag{12}$$

Here and throughout this work we shall be concerned with the evolution equation of the lowest order term in the perturbation expansion. The use of the conventional reductive perturbation method to study the effects of higher order perturbation expansion leads to secularities in the solution. In order to remove such secularities the modified form of the reductive perturbation method [10] or multiple scale expansion method [21] must be utilized.

2.1 Solution of the field equations

Nothing that the equation (10) is linear in u_1, we shall seek a harmonic wave solution to this equation of the following form

$$u_1 = U(\xi,\tau)e^{i\varphi} + c.c. \tag{13}$$

where $U(\xi,\tau)$ is an unknown function whose governing equation will be obtained later, $\varphi = kx - \omega t$ is the phase of harmonic wave and c.c. stands for the complex conjugate of the corresponding quantity. For this order of solution the variables ξ and τ remain as some parameters.

Introducing (13) into (10) and keeping in mind that the dispersion relation (6) holds true, the equation (10) will be satisfied identically. To obtain the solution for $O(\epsilon^4)$ equation we introduce (13) into (11), which results in

$$\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^4 u_2}{\partial x^2 \partial t^2} + 2i\left[\omega\lambda(1+k^2) - k(1-\omega^2)\right]\frac{\partial U}{\partial \xi}e^{i\varphi}$$
$$+4\omega k|U|^2U^2e^{i2\varphi} + 4\omega k U^4 e^{4i\varphi} + c.c. = 0. \tag{14}$$

where $|U|^2 = UU^*$, U^* being the complex conjugate of U. Here it is to be noted that the differential equation (14) is linear in u_2 but the non-homogeneous part is nonlinear in $U(\xi,\tau)$.

The form of equation (14) suggests us that for this order of equation the total solution should have the following form

$$u_2 = U_2^{(1)}e^{i\varphi} + U_2^{(2)}e^{2i\varphi} + U_2^{(4)}e^{i4\varphi} + c.c. \quad . \tag{15}$$

Here $U_2^{(1)}$ is an arbitrary function of its argument and it corresponds to the solution of homogeneous equation. The remaining parts of (15) are related to the particular integral of (14). Introducing (15) into (14) one obtains

$$2i\Big[\omega\lambda(1+k^2) - k(1-\omega^2)\Big]\frac{\partial U}{\partial \xi}e^{i\varphi} + \Big(-12\omega^2k^2U_2^{(2)} + 4\omega k|U|^2U^2\Big)e^{i2\varphi}$$
$$+\Big(-240\omega^2k^2U_2^{(4)} + 4\omega k U^4\Big)e^{i4\varphi} + c.c. = 0. \tag{16}$$

In order to have a non-zero solution for U the coefficient of $\partial U/\partial\xi$ must vanish

$$\omega\lambda(1+k^2) - k(1-\omega^2) = 0, \quad \text{or,} \quad \lambda = \frac{k(1-\omega^2)}{\omega(1+k^2)}. \tag{17}$$

where $\lambda = d\omega/dk$ is the group velocity of the harmonic wave derived from the dispersion relation (6). From the solution of the remaining part of equation (16) we obtain

$$U_2^{(2)} = \frac{|U|^2 U^2}{3\omega k}, \quad U_2^{(4)} = \frac{U^4}{60\omega k}. \tag{18}$$

For the solution of $O(\epsilon^7)$ equation, introducing the solutions given in (13) and (15) into (12) the following equation is obtained

$$\frac{\partial^2 u_3}{\partial t^2} - \frac{\partial^2 u_3}{\partial x^2} - \frac{\partial^4 u_3}{\partial x^2 \partial t^2} + \Big\{ -2i\omega(1+k^2)\frac{\partial U}{\partial \tau} + \Big[\omega^2 + \lambda^2(1+k^2)$$

$$+4\lambda\omega k - 1\Big]\frac{\partial^2 U}{\partial \xi^2} + k\omega\Big(U^3 U_2^{*(2)} + 3|U|^2 U^* U_2^{(2)} + U^{*3} U_2^{(4)}\Big)\Big\} e^{i\varphi}$$

$$+2i\Big[\lambda\omega(1+k^2) - k(1-\omega^2)\Big]\frac{\partial U_2^{(1)}}{\partial \xi} e^{i\varphi} + \sum_{l=2}^{7} f_l e^{il\varphi} + c.c. = 0, \tag{19}$$

where the functions $f_l (l = 2, ..., 7)$ can be expressed in terms of U and $U_2^{(1)}$. But in order to save the space we shall not list them here. Here we note that due to the equation (16) the coefficient of $\partial U_2^{(1)}/\partial \xi$ vanishes.

The form of equation (19) suggests us that the solution of u_3 should have the following form

$$u_3 = \sum_{l=1}^{7} U_3^{(l)} e^{il\varphi} + c.c. \tag{20}$$

Inserting (20) into (19) and considering the dispersion relation (6) the following equation is obtained

$$\Big\{ -2i\omega(1+k^2)\frac{\partial U}{\partial \tau} + \Big[\omega^2 + \lambda^2(1+k^2) + 4\lambda\omega k - 1\Big]\frac{\partial^2 U}{\partial \xi^2} + k\omega\Big(U^3 U_2^{*(2)}$$

$$+3|U|^2 U^* U_2^{(2)} + U^{*3} U_2^{(4)}\Big)\Big\} e^{i\varphi} + \sum_{l=2}^{7} \Big[f_l - l^2\omega^2 k^2 (l^2-1) U_3^{(l)}\Big] e^{il\varphi} + c.c. = 0. \tag{21}$$

Setting the coefficients of various powers of $\exp(i\varphi)$ equal to zero yields the following equations

$$-2i\omega(1+k^2)\frac{\partial U}{\partial \tau} + \Big[\omega^2 + \lambda^2(1+k^2) + 4\lambda\omega k - 1\Big]\frac{\partial^2 U}{\partial \xi^2}$$

$$+k\omega\Big(U^3 U_2^{*(2)} + 3|U|^2 U^* U_2^{(2)} + U^{*3} U_2^{(4)}\Big) = 0, \tag{22}$$

$$f_l - l^2\omega^2 k^2 (l^2-1) U_3^{(l)} = 0, \quad (l = 2, 3, ..., 7). \tag{23}$$

Introducing the expressions of $U_2^{(2)}$ and $U_2^{(4)}$ into (22) the following evolution equation is obtained

$$i\frac{\partial U}{\partial \tau} - \mu_1 \frac{\partial^2 U}{\partial \xi^2} - \mu_2 |U|^6 U = 0, \tag{24}$$

where the coefficients μ_1 and μ_2 are defined by

$$\mu_1 = \frac{\omega^2 + \lambda^2(1+k^2) + 4\lambda\omega k - 1}{2\omega(1+k^2)}, \quad \mu_2 = \frac{27}{40\omega(1+k^2)}. \tag{25}$$

The evolution equation (24) is known as the non-linear Schrödinger equation with seventh order of nonlinearity. Here we note that the order of nonlinearity of the original symmetric regularized long-wave equation is 4 whereas the order of the corresponding nonlinear Schrödinger equation is 7. The equation (23) makes it possible to express $U_3^{(l)}$ in terms of U and $U_2^{(1)}$ which might be used in higher order perturbation expansion.

2.2 Progressive wave solution

The form of the progressive wave solution of the nonlinear Schrödinger equation depends on the sign of the product of coefficients $\mu_1\mu_2$. As is seen from equation (25) this product is positive for all positive wave numbers. In this sub-section we shall seek a progressive wave solution to the evolution equation (24) of the form

$$U = f(\zeta)\exp[i(K\xi - \Omega\tau)], \quad \zeta = \xi + 2\mu_1 K\tau, \tag{26}$$

where $f(\zeta)$ is a real function, K and Ω are some constants. Introducing (26) into (24) one has

$$\mu_1 f'' - (\Omega + \mu_1 K^2) f + \mu_2 f^7 = 0. \tag{27}$$

Here a prime denotes the differentiation of the corresponding quantity with respect to ζ. Since the coefficients μ_1 and μ_2 satisfy the inequality $\mu_1\mu_2 > 0$, the solution for $f(\zeta)$ may be given by

$$f(\zeta) = a \,\mathrm{sech}^{1/3}\beta\zeta \tag{28}$$

where a is the amplitude of the solitary wave, Ω and β are defined by

$$\beta^2 = \frac{9\mu_2 a^6}{4\mu_1}, \quad \Omega = \frac{\mu_2}{4}a^6 - \mu_1 K^2. \tag{29}$$

This shows that the NLS equation with seventh order of nonlinearity also assumes the envelope solitary wave solution as given in (28). One should also note that the frequency of the harmonic wave is proportional to the sixth power of the solitary wave amplitude.

3 Conclusion

As pointed out before, the conventional reductive perturbation method cannot be used to study amplitude modulation of nonlinear waves when the order of nonlinearity is greater than three. In this work, by utilizing the classical reductive perturbation method and a new set of slow variables, we have studied the amplitude modulation of the symmetric regularized long- wave equation with quartic nonlinearity and obtained the nonlinear Schrödinger equation with seventh order of nonlinearity, as the evolution equation. It is observed that although the order of nonlinearity of original equation is four the order of the resulting evolution equation is seven. By seeking a progressive wave solution to the evolution equation we have determined the speeds of the harmonic and envelope waves. It is further observed that the envelope wave is still a solitary wave and the frequency of harmonic wave is proportional to the six power of the wave amplitude.

4 References

1. Debnath, L. (1994) Nonlinear Water Waves, Academic Press Inc. USA.

2. Johnson, R. S. (1997) A Modern Introduction to the Mathematical Theory of Water Waves, Cambridge University Press, U.K.

3. Colin, T., Dias, F. and Ghidagliu(1995), On the rotational effect in the modulation of weakly nonlinear water waves over finite depth, European J. Mech. B- Fluids 14:775-793.

4. Dullin, H. R., Gottwald and Holm, D. D.(2001), An integrable shallow water equation with linear and nonlinear dispersion, Phys. Rev. Lett.

87:4501-4504.

5. Kliakhandler, I. and Trulsen, K. (2000) On weakly nonlinear modulation of waves in deep water, Phys. Fluids, 12: 2432-2437.

6. Abourabia, A. M., Hassan, K. M. and Selima, E. S. (2010), The derivation and study of the nonlinear Schrödinger equation for long waves in shallow water using the reductive perturbation method and complex ansatz methods, Int. J. Nonlinear Sci., 9:430-443.

7. Jukui, X.(2003) Modulational instability of ion acoustic waves in a plasma consisting of warm ions and non-thermal electrons , Chaos, Solitons and Fractals, 18:849-853.

8. Xue, J. (2004) Cylindrical and spherical ion-acoustic solitary waves with dissipative effects, Phys. Rev. Lett. A, 322:225-230.

9. Gill, S., Kaur, H. and Saini, N. S.(2006) Small amplitude electron-acoustic solitary waves in a plasma with non-thermal electrons, Chaos, Solitons and Fractals, 30:1020-1024.

10. Demiray, H.(2002) Contribution of higher order terms in non-linear ion-acoustic waves: Strongly dispersive case, J. Phys. Soc. Japan, 71:1921-1930.

11. El-Labany, S. K. (1991) Modulation of ion-acoustic waves, Astrophysics and Space Science, 182:241-247.

12. Ravindran, R. and Prasad, P.(1979) A mathematical analysis of nonlinear waves in a fluid-filled viscoelastic tube, Acta Mechanica, 31:253-280.

13. Erbay, H. A. and Erbay, S.(1994) Nonlinear wave modulation in a fluid-filled distensible tube, Acta Mechanica, 104: 201-214.

14. Demiray, H.(2001) Modulation of non-linear waves in a viscous fluid contained in an elastic tube, Int. J. Nonlinear Mech., 36:649-661.

15. Demiray, H.(2003), On the contribution of higher order terms to solitary waves in fluid-filled elastic tubes, Int. J. Engr. Sci., 41: 1387-1403.

16. Demiray, H. (2005) Higher order approximation in reductive perturbation method: Strongly dispersive waves, Communication in Nonlinear Sci. and Numerical Simulation, 10: 549-558.

17. Benjamin,T., Bonna, J. and Mahoney, J. (1978) Model equations for long waves in nonlinear dispersive systems, Trans. Roy. Soc. London Ser.A,

272:47-78.

18. Seyler, C. E. and Fenstermacher, D. L.(1984) A symmetrical regularized long-wave equation, Physics of Fluids, 27: 4-7.

19. Chen, L.(1998) Stability and instability of solitary waves for generalized symmetric regularized-long-wave equations, Physica D, 118: 53-68.

20. Yong, C. and Biao, L.(2004) Travelling wave solutions for generalized symmetric regularized long-wave equations with high-order nonlinear terms, Chinese Physics, 13:302-306.

21. Demiray, H.(2010) Multiple time scale formalism and its applications to long water waves, Appl. Math. Modelling, 34:1187-1193.